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ON THE STRUCTURE OF SUBSONIC FLOW BETWEEN AN ASYMMETRIC BODY AND A DETACHED SHOCK WAVE†

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Subsonic vortex flow behind a detached shock wave (SW) that arises during supersonic plane flow around an asymmetric body of finite thickness is considered. For bodies with convex leading edges it is shown that the subsonic section of the SW is convex, and that there are no internal branch points in the region between the body, the SW and the isobars emerging from the sonic points of the SW. It is shown that this region is covered by the family of isoclines emerging from the stagnation point. This convexity of the subsonic section of the SW is also possible for a non-convex body provided the values of the angles of inclination of the walls at the leading edge of the body lie outside the range of shock polar angles.

THE ISOBAR method, based on analysing lines of constant pressure [1–4], and the similar "modified" hodograph method [5] have been used previously for symmetric flow problems. Both the isobar method and a method based on analysing lines of constant angle of inclination of the velocity vector (the isocline method) turned out to be necessary for investigating asymmetric flow. Both methods are based on the well-known properties of isobars and isoclines [1, 6].

1. We will consider some relations and concepts which will be used later.

Plane vortex flow of an ideal (non-viscous and non-heat-conducting) gas is described by the equations [1]

$$\rho q^2\theta_L = -p_N, \quad \rho q^2\theta_N = -p_L(M^2-1) \tag{1.1}$$

where p and ρ are the pressure and density, q and θ are the absolute value and angle of inclination of the velocity vector, M is the Mach number, and θ_L, p_L, θ_N and p_N are derivatives taken along the streamlines and along the normals to them.

As corollaries of Eqs (1.1) we have expressions for the derivatives θ_l and p_l computed along the lines $p=\text{const}$ and $\theta=\text{const}$, respectively [1]

$$\theta_l = -p_n(1-M^2\sin^2\beta)/(\rho q^2) \tag{1.2}$$

$$p_l = \theta_n\rho q^2(1-M^2\sin^2\varphi)/(1-M^2) \tag{1.3}$$

where p_n and θ_n are derivatives computed along the normals to the lines $p=\text{const}$ and $\theta=\text{const}$, respectively, and β and θ are the angles between these lines and the velocity vector.

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Henceforth an isobar (isocline) is taken to be a line $p = \text{const}$ ($\theta = \text{const}$) which bounds a region of greater or lower pressure (the angle θ) than that on the line under consideration. From relations (1.2) and (1.3) and using the above we find that in subsonic vortex flows the angle θ (the pressure p) changes monotonically along the isobars (isoclines), but not necessarily strictly monotonically [1]. The equalities $\theta_l = p_n = p_l = \theta_n = 0$ are only possible at isolated branch points at which all first derivatives of p and θ vanish.

It can be shown that for subsonic flows an equal even number $N \geqslant 4$ of isobars and isoclines emerge from isolated branch points, with exactly one isocline lying between two neighbouring isobars and vice versa.

Indeed, consider a circle with centre at the branch point w and of sufficiently small radius for there to be no other branch point inside it. It is obvious that in going along this circle around the point w, one encounters an isobar on which the derivative $p_n > 0$ after any isobar on which $p_n < 0$, etc. Consequently, an even number N of isobars must emerge from the point w. For an ordinary point that is not a branch point $N = 2$, while for a branch point $N \geqslant 4$.

Consider two neighbouring isobars emerging from the point w. Along one of these θ increases, while along the other it decreases. Consequently, between these isobars there is an isocline, also emerging from the point w. There are no other similar isoclines between the two isobars under consideration, as otherwise one could show that between the two neighbouring isoclines there was an isobar also emerging from the branch point, which would contradict the original assumption. The assertion is proved.

We note that there is also a completely different type of branch point at the point w if at that point there is not one, but a whole range of values of θ. In this case, an infinite number of isoclines emerge from the point w, there is a family of closed isobars around this point, and, as a consequence, no isobars emerge from it into the region under consideration. Such situations occur at internal stagnation points and in flows with closed streamlines.

Figure 1 shows the shock polar for the uniform supersonic flow of a polytropic gas. The polar is symmetrical about the $\theta = 0$ axis. The points k and k^- correspond to the largest value $\theta = \theta_k$ and the smallest value $\theta = -\theta_k$ of the angle θ. At the points c and c^- we have $M = 1$, and above (below) the points c and c^- we have $M < 1$ ($M > 1$). For the polytropic gas under consideration the points c and c^- lie below the points k and k^-.

We will list some known properties of isobars and isoclines in the subsonic region matched across the SW to uniform supersonic flow.

1. There are no closed isoclines, and when there are no inner stagnation points and closed streamlines there are also no closed isobars.

2. There are no isobars with both end points lying on the SW [1]. The exceptions are isobars surrounding a point at which $\theta = 0$ and at which the SW is convex towards the region of the subsonic flow beyond the SW [2].

3. There are no isobars with end points on a straight wall if the wall between those points does not have a stagnation point [1,3].

We also state the following properties of isoclines that have not been noted previously.

4. As isocline cannot start and end on an SW, except in cases when it surrounds a point on the SW where $\theta = \pm\theta_k$ and in a neighbourhood of which the acceleration is negative.

The proof is based on the properties of shock polars and the monotonic variation of p along an isocline.

5. An isocline cannot begin and end on a straight-line wall segment.

To show this we note that the wall segment lying between end points, when there are no branch points on it, is also an isocline. But closed isoclines cannot exist. If there is a branch point then an isocline will emerge from it into the flow, and this isocline cannot intersect the original isocline and cannot return to the wall because we would again have a closed isocline.

These properties of isoclines and isobars will be used below.

2. We will consider plane flow around an asymmetric body with convex leading edge and uniform horizontal supersonic flow incident from the left, Fig. 2. Here b^-b is the profile of the body z^-z is the detached shock wave (SW) c^-a^- and ca are sonic lines, tO is the separating

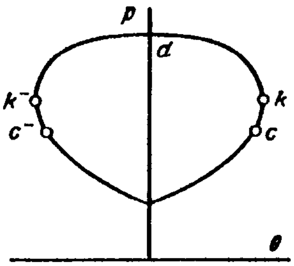

Fig. 1.

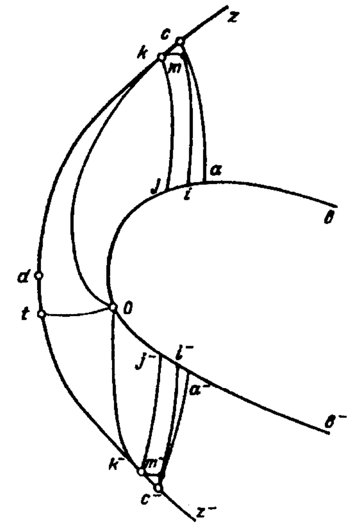

Fig. 2.

line, and O is the stagnation point.

The leading edge of the body is assumed to be sufficiently smooth and without corners, so that in the c^-a^- and ac region there are no local supersonic regions or closed streamlines. It is obvious that there are no internal stagnation points in this region.

Below we are not fundamentally concerned with investigating the entire subsonic region c^-a^-Oac, but a large portion of it: c^-a^-Oic, where c^-i^- and ci are isobars. Using the fact that the entropy at the points c^- and c does not exceed the value of the entropy at any point of the subsonic section of the SW, one can show that at c^-i^- and ci $M \leqslant 1$ [2]. Consequently, in accordance with the above, the value of θ decreases monotonically along c^-i^- while along ci it increases.

Theorem 1. Suppose that the segment i^-i of the body being considered is convex, i.e. when moving from i^- to i the angle of inclination of the wall decreases monotonically (though not necessarily strictly monotonically), so that straight-line segments are possible. Then the subsonic flow between the body and the SW has the following properties.

1. When moving along the SW from the point c^- to the point d, at which $\theta = 0$, the pressure increases monotonically, and then falls along the segment dc. As a corollary, the segment c^-c is convex.

2. In the domain c^-i^-ic there are no inner branch points for isobars or isoclines.

Proof. Consider the behaviour of θ along the contour c^-i^-Oic using the known solution for an infinitesimally small neighbourhood of a stagnation point [7]. θ diminishes along the c^-i^-O part of the contour, possibly non-strictly monotonically along i^-O (in the presence of straight-line wall segments). At the point O, θ increases from $\theta = \theta_0 - \pi$ to $\theta = \theta_0$, and for each value of θ in this range there is one isocline, making an angle $\omega = 2\theta_0 - \theta$ with the x axis at the point O. Finally, along the contour Oic, θ again decreases, and along the wall Oi straight-line sections are possible along which $\theta = \text{const}$. This behaviour of θ along c^-i^-Oic plays an important role in later arguments.

Along the subsonic c^-c part of the SW p and θ which are related by the shock polar, change continuously. Hence there is at least one point on c^-c at which $\theta = \theta_k$, and in a neighbourhood of which the derivative p_τ, computed along c^-c is negative. In accordance with the above-mentioned isocline properties this point is not surrounded by isoclines starting and ending at the SW. Taking into account that at this point the derivative $\theta_\tau = 0$, we conclude that at least two isoclines emerge from it into the subsonic region. These isoclines cannot go outside the subsonic section of the SW, and cannot terminate inside the subsonic flow domain. They can only reach the contour c^-i^-Oic. Even if the value of $\theta=\theta_k$ on this contour corresponds to a linear segment of the wall, then, in accordance with the properties of isoclines, only one isocline can run from the point being considered to this segment. Consequently, on the subsonic part of the SW there is only one point k at which $\theta=\theta_k$ and in a neighbourhood of which $p_\tau<0$, and two isoclines emerge from this point, one of which arrives at the stagnation point O, and the other at the contour Oic. (We are in fact referring to a single isocline tangential to the SW at the point k.)

The point k^- is defined similarly. Both points and the isoclines Okm and Ok^-m^- are shown in Fig. 2. Below it will be of absolutely no relevance whether the points m and m^- lie on isobars, as shown in Fig. 2, or on the body. In either case, θ changes monotonically along the sections mc and m^-c^-.

Between the points k and k^- on the SW we have $p\geqslant p_k$. Consequently, using the above, isoclines emerging from points along the section k^-k can only arrive at the stagnation point O, which in turn proves the monotonic increase in θ along the section k^-k, the monotonic increase of p along the section k^-d, and the monotonic decrease of p along the section dk, and, as a corollary, the convexity of the SW along the section k^-k.

To the right of the point k with p varying monotonically, isoclines that begin and end on the section kc can exist. Hence the isocline method used above does not work when investigating the section kc. The isobar method, used previously for symmetric flow problems [1–4], is more appropriate for it.

We will show that only one isobar emerges from the point k. If we assume otherwise, with $p_\tau=0$ at the point k, at least three isobars emerge from this point. When moving from the point k along extreme isobars we have $p_n<0$, and consequently θ increases. In these discussions it is assumed that $p_r<0$ in the neighbourhood of k. Then extreme isobars, along which $\theta>\theta_k$, can arrive only at the wall Oi. Between these isobars there is at least one isobar along which θ decreases, and it should also arrive at the convex wall Oi which is ruled out. A similar argument applies to the point k^-.

Thus, from the points k and k^- the isobars kj and k^-j^- emerge, and at the points j and j^- we have $\theta>\theta_k$ and $\theta<-\theta_k$, respectively. From these inequalities we also have an estimate for the angle of inclination of the wall at the stagnation point

$$\theta_k<\theta_0<\pi-\theta_k$$

We shall assume that p varies non-monotonically along the section kc. Then one can choose points f and g on kc with equal values of p and θ but with different values of p_τ. At the point g, situated on the right, $p_\tau<0$, and at the point f, $p_\tau>0$. As a result θ increases along the isobar emerging from the point g, and decreases along the isobar emerging from the point f, and that isobar arrives at the body to the right of the point j, i.e. it cannot arrive at the section Oi^- at which $\theta<0$. But for a convex body this situation is ruled out. A similar argument applies to the section k^-c^-, which completes the proof of the first assertion of the theorem.

From the preceding arguments it is clear that there are no branch points on the isoclines Okm and Ok^-m^-. This property enables one to demonstrate the absence of branch points inside the domain c^-i^-ic separately for the domains kcm, $k^-c^-m^-$, $Okmi$, $Ok^-m^-i^-$, and Okk^-.

Suppose, for example, that some inner point of the domain $Okmi$ is a branch point. Then at least four isobars and four isoclines emerge from it. The isoclines emerging from the branch point cannot reach the isocline Okm because the latter has no branch points. But on the contour Oim, even taking into account any possible straight-line wall sections, and at the stagnation point only two out of the four (or more) isoclines leaving the branch point can arrive. Consequently, from any inner point of the domain $Okmi$ only two isoclines, and consequently also two isobars, can leave, or equivalently, only one isocline and one isobar can pass through every inner point of the domain $Okmi$.

One can similarly show the absence of branch points in the domains $Ok^-m^-i^-$, Ok^-k, kmc, $k^-c^-m^-$, which completes the proof of the theorem.

Corollary 1. There are no closed isobars or isoclines in the domain c^-i^-ic, and as a corollary, there are no internal local extrema for p and θ. Consequently, the second assertion of Theorem 1 shows that the derivatives p_x, p_y, θ_x, θ_y cannot simultaneously vanish at any inner point of the domain under consideration.

We also note that the results obtained, like the results in [4] referring to subsonic symmetric flow around a convex body, demonstrate the relationship between boundary conditions and flow properties such as the absence of internal branch points.

Corollary 2. The flow domain $imkk^-m^-i^-$ is completely covered by the family of isoclines emerging from the stagnation point O. Along each isocline the pressure decreases monotonically.

As in the symmetric case [3], the convexity of the leading edge of the body is a sufficient, but not a necessary condition for convexity of the subsonic section of the SW. We have the following theorem.

Theorem 2. Let $\theta > \theta_k$ and $\theta < -\theta_k$ on the sections Oi and Oi^-, respectively, on a not necessarily convex body (Fig. 2). (This condition is satisfied, for example, if on the right boundary of the leading edge there are convex angular points to the left of which $\theta > \theta_k$ and $\theta < -\theta_k$, respectively, above and below the stagnation point. In this case, depending on the shape of the body, the sonic lines emerge from the angular points or from points to their left.) When these conditions are satisfied on the subsonic part of the SW, the pressure p increases (decreases) monotonically along c^-d (dc) and, as a corollary, the subsonic part of the SW c^-c is convex.

The proof is almost a repeat of the proof of the first assertion of Theorem 1. In the analysis of the segments kc and k^-c^- the required contradiction is obtained by considering isobars emerging from points with the assumed negative acceleration.

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